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COHOMOLOGY OF THE MUMFORD QUOTIENT

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Abstract. Let X be a smooth projective variety acted on by a reductive group G. Let L be a positive G-equivariant line bundle over X. We use a Witten type deformation of the Dolbeault complex of L, introduced by Tian and Zhang, to show, that the cohomology of the sheaf of holomorphic sections of the induced bundle on the Mumford quotient of (X, L) is equal to the G-invariant part on the cohomology of the sheaf of holomorphic sections of L. This result, which was recently proven by C. Teleman by a completely different method, generalizes a theorem of Guillemin and Sternberg, which addressed the global sections. It also shows, that the Morse-type inequalities of Tian and Zhang [11] for symplectic reduction are, in fact, equalities.

1. Introduction

1.1. **Cohomology of the set of semistable points.** Suppose X is a smooth complex projective variety endowed with a holomorphic action of a complex reductive group G. Given a positive G-equivariant holomorphic line bundle L over X, Mumford [7] defined the notion of stability: A point $x \in X$ is called *semi-stable* with respect to L if and only if there exist a positive integer $m \in \mathbb{N}$ and an invariant section $s : X \to L^m$ such that $s(x) \neq 0$. Let X^{ss} denote the set of semi-stable points of X. It is a G-invariant Zariski open subset of X. Let L^{ss} denote the restriction of L to X^{ss}.

The group G acts naturally on the cohomology $H^*(X, \mathcal{O}(L))$ and $H^*(X^{ss}, \mathcal{O}(L^{ss}))$ of the sheaves of algebraic sections of L and L^{ss} respectively. We denote by $H^*(X, \mathcal{O}(L))^G$ and $H^*(X^{ss}, \mathcal{O}(L^{ss}))^G$ the subspaces of G-invariant elements in $H^*(X, \mathcal{O}(L))$ and $H^*(X^{ss}, \mathcal{O}(L^{ss}))$.

The main result of this paper is the following

Theorem 1.1. *Assume that the action of G on X^{ss} is free. Then*

$$\dim_{\mathbb{C}} H^j(X, \mathcal{O}(L))^G \;=\; \dim_{\mathbb{C}} H^j(X^{ss}, \mathcal{O}(L^{ss}))^G, \qquad j = 0, 1, \dots \tag{1.1}$$

Remark 1.2. The equality (1.1), implies that there is an isomorphism between vector spaces $H^j(X, \mathcal{O}(L))^G$ and $H^j(X^{ss}, \mathcal{O}(L^{ss}))^G$. We actually construct a family of such isomorphisms in the course of the proof. Unfortunately, we don't have a way to single out one canonical isomorphism.

1.2. **Historical remarks.** For $j = 0$, the equality (1.1) was established by Guillemin and Sternberg [3, §5]. Guillemin and Sternberg also conjectured that

$$\sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(X, \mathcal{O}(L))^G \;=\; \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(X^{ss}, \mathcal{O}(L^{ss}))^G. \tag{1.2}$$

This was proven by Meinrenken in [5]. An analytic proof of (1.2) was given by Tian and Zhang [11, Th. 0.4], who also proved the inequality

$$\dim_{\mathbb{C}} H^j(X, \mathcal{O}(L))^G \leq \dim_{\mathbb{C}} H^j(X^{ss}, \mathcal{O}(L^{ss}))^G. \tag{1.3}$$

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Theorem 1.1 may be considered as an amplification of the results of Guillemin-Sternberg, Meinrenken and Tian-Zhang.

The equality (1.1) (with somehow weaker assumptions on the action of G on X) was recently proven by C. Teleman [10]. His proof is completely algebraic. It is based on the study of the sheaf cohomology of $\mathcal{O}(L)$ with support on a stratum of the Morse stratification for the square of the moment map (see also [8], where a similar technique was used).

The purpose of this paper is to give a direct analytic proof of Theorem 1.4, based on the study of the Witten type deformation of the Dolbeault complex of X with values in L, introduced by Tian and Zhang [11].

Certain generalizations of (1.2) for "singular reduction", i.e., when the action of G on X^{ss} is not free, was obtained by Sjamaar [9], Meinrenken and Sjamaar [6]. Tian and Zhang [11] gave a new analytic proof of these results and also extended the inequalities (1.3) to the singular reduction. Recently, Zhang [12] used the method of this paper to extend (1.1) to singular reduction.

1.3. **The cohomology of the Mumford quotient.** Suppose again that the action of G on the set X^{ss} of semi-stable points is free. Then the quotient space X^{ss}/G is a smooth projective variety, called the *Mumford quotient* of X. The quotient L^{ss}/G has a natural structure of a holomorphic line bundle over X^{ss}/G. Clearly, the quotient map $q : X^{ss} \to X^{ss}/G$ induces a natural isomorphism

$$H^j(X^{ss}/G, \mathcal{O}(L^{ss}/G)) \simeq H^j(X^{ss}, \mathcal{O}(L^{ss}))^G, \quad j = 0, 1, \dots \tag{1.4}$$

Hence, Theorem 1.1 is equivalent to the following

Theorem 1.3. *Assume that the action of G on X^{ss} is free. Then*

$$\dim_{\mathbb{C}} H^j(X, \mathcal{O}(L))^G = \dim_{\mathbb{C}} H^j(X^{ss}/G, \mathcal{O}(L^{ss}/G)), \qquad j = 0, 1, \dots \tag{1.5}$$

1.4. **Reformulation in terms of symplectic reduction.** Let $K \subset G$ be a maximal compact subgroup of G and let $\mathfrak{k} = Lie(K)$ denote the Lie algebra of K. Since, G is a complexification of K, it is clear, that the space of G-invariant elements of $H^*(X, \mathcal{O}(L))$ coincides with the space of K-invariant elements:

$$H^j(X, \mathcal{O}(L))^G = H^j(X, \mathcal{O}(L))^K, \qquad j = 0, 1, \dots \tag{1.6}$$

Fix a K-invariant hermitian structure on L and let ∇ be a K-invariant holomorphic hermitian connection on L. Denote

$$\omega = \frac{i}{2\pi} (\nabla)^2.$$

Then ω is a K-invariant Kähler form on X, representing the Chern class of L in the integer cohomology of X.

For any section $s : X \to L$ and any $V \in \mathfrak{k}$, we denote by $\mathcal{L}_V s$ the infinitesimal action of V on s, induced by the action of $K \subset G$ on L.

Define a map $\mu : X \to \mathfrak{k}^*$ by the formula

$$\langle \mu, V \rangle = \frac{i}{2\pi} (\mathcal{L}_V - \nabla_V), \qquad V \in \mathfrak{k}.$$

Here ∇_V denotes the covariant derivative along the vector field generated by V on X. Then, cf. [4], μ is a *moment map* for the action of K on the symplectic manifold (X, ω).

The assumption that G acts freely on X^{ss} is equivalent to the statement that $0 \in \mathfrak{k}^*$ is a regular value of μ and K acts freely on $\mu^{-1}(0)$, cf. [3, §4]. Let $X_0 = \mu^{-1}(0)/K$ denote the symplectic reduction of X at 0. Then, cf. [3, §5], X_0 is complex isomorphic to the Mumford quotient:

$$X_0 = X^{ss}/G. \tag{1.7}$$

Let $\overline{q} : \mu^{-1}(0) \to \mu^{-1}(0)/K$ denote the quotient map. Then, cf. [3, Th. 3.2], there exists a unique holomorphic line bundle L_0 on X_0 such that $\overline{q}^*L_0 = L|_{\mu^{-1}(0)}$. Moreover, under the isomorphism (1.7), L_0 is isomorphic to L^{ss}/G. Hence, it follows from (1.6), that Theorems 1.1, 1.3 are equivalent to the following

Theorem 1.4. *Suppose* 0 *is a regular value of* μ *and* K *acts freely on* $\mu^{-1}(0)$*. Then*

$$\dim_{\mathbb{C}} H^j(X, \mathcal{O}(L))^K \;=\; \dim_{\mathbb{C}} H^j(X_0, \mathcal{O}(L_0)), \qquad j = 0, 1, \dots \tag{1.8}$$

The proof of Theorem 1.4 is given in Subsection 3.2. Here we will only explain the main idea of the proof.

1.5. **A sketch of the proof.** Following Tian and Zhang [11], we consider the one parameter family $\bar\partial_t = e^{-t\frac{|\mu|^2}{2}}\bar\partial e^{t\frac{|\mu|^2}{2}}$ of differentials on the Dolbeault complex $\Omega^{0,*}(X, L)$. Let $\bar\Box_t = \bar\partial_t\bar\partial_t^* + \bar\partial_t^*\bar\partial_t$ denote the corresponding family of Laplacians and let $\bar\Box_t^{j,K}$ denote the restriction of $\bar\Box_t$ to the space of K-invariant forms in $\Omega^{0,j}(X, L)$. Let $E_{t,\lambda}^{j,K}$ denote the span of eigenforms of $\bar\Box_t^{j,K}$ with eigenvalues smaller than λ. Then $H^*(X, \mathcal{O}(L))^K$ is isomorphic to the cohomology of the complex $(E_{t,\lambda}^{*,K}, \bar\partial_t)$.

In [11], Tian and Zhang showed that there exists $\lambda > 0$ such that

$$\dim H^j(X_0, \mathcal{O}(L_0)) = \dim E_{t,\lambda}^{j,K}, \qquad j = 0, 1, \dots$$

for any $t \gg 0$. Moreover, if we denote by $\mathcal{H}^j(X_0, L_0) \subset \Omega^{0,j}(X_0, L_0)$ the subset of harmonic forms (with respect to the metrics introduced in Subsection 2.3), Tian and Zhang constructed an explicit isomorphism of vector spaces

$$\Phi_{\lambda,t}^j : \mathcal{H}^j(X_0, L_0) \to E_{t,\lambda}^{j,K}.$$

Recall that we denote by q the quotient map $X^{ss} \to X_0 = X^{ss}/G$. In Section 3, we consider the integration map $I_t : \Omega^{0,j}(X, L) \to \Omega^{0,j}(X_0, L_0)$ defined by the formula [1]

$$I_t : \alpha \;\mapsto\; \left(\frac{t}{2\pi}\right)^{r/4} \int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}} \alpha \wedge \omega^r,$$

where $r = \dim_{\mathbb{C}} G$. Then we show (cf. Theorem 3.1) that $I_t\bar\partial_t = \bar\partial I_t$, for any $t \geq 0$. Also the restriction of I_t to $E_{\lambda,t}^{*,K}$ is "almost equal" to $(\Phi_{\lambda,t}^*)^{-1}$ for $t \gg 0$.

It follows from the existence of the map I_t with the above properties, that the restriction of $\bar\partial_t$ to $E_{\lambda,t}^{*,K}$ vanishes (cf. Corollary 3.2). This implies Theorem 1.4 and, hence, Theorems 1.1 and 1.3.

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2. Witten type deformation of the Dolbeault complex. The Tian-Zhang theorem

Our proof of Theorem 1.4 uses the Witten type deformation of the Dolbeault complex, introduced by Tian and Zhang [11]. For convenience of the reader, we review in this section the results of Tian and Zhang which will be used in the subsequent sections.

[1] The map I is defined as an integral over the fibers of q. One can think about I as a kind of equivariant push-forward of differential forms under q.

2.1. **Deformation of the Doulbeault complex.** Let $\Omega^{0,j}(X,L)$ denote the space of smooth $(0,j)$-differential forms on X with values in L. The group K acts on this space and we denote by $\Omega^{0,j}(X,L)^K$ the space of K-invariant elements in $\Omega^{0,j}(X,L)$.

Recall that $\mu : X \to \mathfrak{k}^*$ denotes the moment map for the K action on X. Let $\mathfrak{k}$ (and, hence, $\mathfrak{k}^*$) be equipped with an Ad K-invariant metric, such that the volume of K with respect to the induced Riemannian metric is equal to 1. Let $h_1, \dots, h_r$ denote an orthonormal basis of $\mathfrak{k}^*$. Denote by v_i the Killing vector fields on X induced by the duals of h_i. Then $\mu = \sum \mu_i h_i$, where each μ_i is a real valued function on X such that $\iota_{v_i}\omega = d\mu_i$. Hence,

$$d\,\frac{|\mu|^2}{2} \;=\; \sum_{i=1}^{r} \mu_i d\mu_i \;=\; \sum_{i=1}^{r} \mu_i \iota_{v_i}\omega. \tag{2.1}$$

Consider the one-parameter family $\bar{\partial}_t,\ t \in \mathbb{R}$ of differentials on $\Omega^{0,j}(X,L)$, defined by the formula

$$\bar{\partial}_t : \alpha \;\mapsto e^{-t\frac{|\mu|^2}{2}}\,\bar{\partial}\, e^{t\frac{|\mu|^2}{2}}\,\alpha \;=\; \bar{\partial}\alpha \;+\; t\sum_{i=1}^{k} \mu_i \bar{\partial}\mu_i \wedge \alpha.$$

Clearly, $\bar{\partial}_t^2 = 0$ and the cohomology of the complex $(\Omega^{0,*}(X,L), \bar{\partial}_t)$ is K-equivariantly isomorphic to $H^*(X, \mathcal{O}(L))$, for any $t \in \mathbb{R}$.

2.2. **The deformed Laplacian.** Fix a K-invariant Kähler metric on X and a K-invariant Hermitian metric on L. Let $\bar{\partial}_t^*$ denote the formal adjoint of $\bar{\partial}_t$ with respect to these metrics and set

$$\bar{\square}_t \;=\; \bar{\partial}_t^*\bar{\partial}_t \;+\; \bar{\partial}_t\bar{\partial}_t^*.$$

Let $\bar{\square}_t^{j,K}$ denote the restriction of $\bar{\square}_t$ to the space $\Omega^{0,j}(X,L)^K$.

For any $j = 0, 1, \dots, \lambda > 0$ and $t > 0$, we define $E_{\lambda,t}^{j,K}$ to be the span of the eigenforms of $\bar{\square}_t^{j,K}$ with eigenvalues less or equal than λ. Then $E_{\lambda,t}^{*,K}$ is a subcomplex of $(\Omega^{0,*}(X,L), \bar{\partial}_t)$. Since $E_{\lambda,t}^{*,K}$ contains the kernel of $\bar{\square}_t$, it follows from the Hodge theory that the cohomology of $E_{\lambda,t}^{*,K}$ is isomorphic to $H^*(X, \mathcal{O}(L))^K$.

The following theorem of Tian and Zhang [11][2] is crucial for our paper:

Theorem 2.1. *There exist $\lambda, t_0 > 0$ such that, for any $j = 0, 1, \dots$ and any $t > t_0$, λ is not in the spectrum of $\bar{\square}_t^{j,K}$ and*

$$\dim E_{\lambda,t}^{j,K} \;=\; \dim H^j(X_0, \mathcal{O}(L_0)). \tag{2.2}$$

As an immediate consequence of Theorem 2.1 and the fact that $H^*(X, \mathcal{O}(L))^K$ is isomorphic to the cohomology of the complex $(E_{\lambda,t}^{*,K}, \bar{\partial}_t)$ we obtain the following inequalities of Tian and Zhang [11, Th. 4.8]:

$$\dim H^j(X, \mathcal{O}(L))^K \;\le\; \dim H^j(X_0, \mathcal{O}(L_0)), \quad j = 0, 1, \dots \tag{2.3}$$

In Subsection 3.2, we will prove that the restriction of $\bar{\partial}_t$ to $E_{\lambda,t}^{*,K}$ vanishes for $t \gg 0$. In other words, the cohomology of the complex $(E_{\lambda,t}^{*,K}, \bar{\partial}_t)$ is isomorphic to $E_{\lambda,t}^{*,K}$. Hence, the inequalities (2.3) are, in fact, equalities. This will complete the proof of Theorem 1.4.

Since we need some of the results obtained by Tian and Zhang during their proof of Theorem 2.1, we now recall briefly the main steps of this proof.

[2]Theorem 2.1 is a combination of Th. 3.13 and §4.d of [11]

2.3. **Main steps of the proof of Theorem 2.1.** The proof of Theorem 2.1 in [11] goes approximately as follows: one shows first, that the eigenforms of $\bar{\Box}_t^{*,K}$ with eigenvalues smaller than λ are concentrated near $\mu^{-1}(0)$. Then, using the local form of $\bar{\Box}_t^{*,K}$ near $\mu^{-1}(0)$, one describes the "asymptotic behaviour" of $E_{\lambda,t}^{*,K}$ as $t\to\infty$.

More precisely, recall that $\overline{q}:\mu^{-1}(0)\to X_0=\mu^{-1}(0)/K$ denote the quotient map and set

$$\widetilde{h}(x) \ = \ \sqrt{\mathrm{Vol}\,\overline{q}^{-1}(x)}.$$

Let g^{L_0} and g^{X_0} denote the Hermitian metric on L_0 and the Riemannian metric on X_0 induced by the fixed metrics on L and X respectively (cf. Subsection 2.2). Set $g^{L_0}_{\widetilde{h}}=\widetilde{h}^2g^{L_0}$ and let $\bar{\partial}^*_{\widetilde{h}}$ denote the formal adjoint of the Dolbeault differential $\bar{\partial}:\Omega^{0,*}(X_0,L_0)\to\Omega^{0,*+1}(X_0,L_0)$ with respect to the metrics $g^{L_0}_{\widetilde{h}},g^{X_0}$. Let

$$\mathcal{H}^j(X_0,L_0) \ = \ \mathrm{Ker}\left(\bar{\partial}\bar{\partial}^*_{\widetilde{h}}+\bar{\partial}^*_{\widetilde{h}}\bar{\partial}:\,\Omega^{0,j}(X_0,L_0)\to\Omega^{0,j}(X_0,L_0)\right)$$

be the space of harmonic forms. Then, for any $t\gg 0$, one constructs an isomorphism of vector spaces

$$\Phi^j_{\lambda,t}:\,\mathcal{H}^j(X_0,L_0)\ \to\ E^{j,K}_{\lambda,t}, \qquad j=0,1,\dots \tag{2.4}$$

This implies the equality (2.2).

Since we will use the above isomorphism in our proof of Theorem 1.4, we now review briefly its construction and main properties.

Remark 2.2. In [11], Tian and Zhang considered the operator

$$D_Q \ = \ \sqrt{2}\,(\,\widetilde{h}\bar{\partial}\widetilde{h}^{-1}+\widetilde{h}^{-1}\bar{\partial}^*\widetilde{h}\,):\,\Omega^{0,j}(X_0,L_0)\ \to\ \Omega^{0,j}(X_0,L_0),$$

where $\bar{\partial}^*$ denotes the formal adjoint of $\bar{\partial}$ with respect to the metrics g^{L_0},g^{X_0}. Then Tian and Zhang used the method of [1] to construct a map from $\mathrm{Ker}\,D_Q$ to $E^{j,K}_{\lambda,t}$. Clearly, $\mathrm{Ker}\,D_Q=\widetilde{h}^{-1}\mathcal{H}^*(X_0,L_0)$. Our map (2.4) is a composition of multiplication by $\widetilde{h}^{-1}$ with the map constructed in [11].

2.4. **A bijection from $\mathcal{H}^*(X_0,L_0)$ onto $E^{*,K}_{\lambda,t}$.** As a first step in the construction of the isomorphism (2.4), we construct an auxiliary map $\Psi^j_t:\Omega^{0,j}(X_0,L_0)\to\Omega^{0,j}(X,L)$.

Let $N\to\mu^{-1}(0)$ denote the normal bundle to $\mu^{-1}(0)$ in X. If $x\in\mu^{-1}(0)$, $Y\in N_x$, let $t\in\mathbb{R}\to y_t=\exp_x(tY)\in X$ be the geodesic in X which is such that $y_0=x$, $dy/dt|_{t=0}=Y$. For $0<\varepsilon<+\infty$, set

$$B_\varepsilon \ = \ \{\,Y\in N:\ |Y|<\varepsilon\,\}.$$

Since X and $\mu^{-1}(0)$ are compact, there exists $\varepsilon_0>0$ such that, for $0<\varepsilon<\varepsilon_0$, the map $(x,Y)\in N\to\exp_x(tY)$ is a diffeomorphism from B_ε to a tubular neighborhood $\mathcal{U}_\varepsilon$ of $\mu^{-1}(0)$ in X. From now on, we will identify B_ε with $\mathcal{U}_\varepsilon$. Also, we will use the notation $y=(x,Y)$ instead of $y=\exp_x(Y)$.

Set $r=\dim K$. Since $0\in\mathfrak{k}^*$ is a regular value of μ, $\mu^{-1}(0)$ is a non-degenerate critical submanifold of $|\mu|^2$ in the sense of Bott. Thus, there exists an equivariant orthonormal base $f_1,\dots,f_r$ of N such that, for any $Y=y_1f_1+\cdots+y_rf_r$,

$$|\mu(x,Y)|^2 \ = \ \sum_{i=1}^r a_iy_i^2 \ + \ O(|Y|^3), \tag{2.5}$$

where each a_i is a positive K-invariant function on $\mu^{-1}(0)$.

Let $p:\mathcal{U}_\varepsilon\to\mathcal{U}_\varepsilon/K$ denote the projection and let h be the smooth positive function on $\mathcal{U}_\varepsilon$ defined by

$$h(u) \ := \ \sqrt{\mathrm{Vol}(p^{-1}(p(u)))}, \quad u\in\mathcal{U}_\varepsilon.$$

Note that, for any $u\in\mu^{-1}(0)$ we have $h(u)=\widetilde{h}(p(u))$, cf. Subsection 2.3.

The following simple lemma plays an important role in Subsection 4.2:

Lemma 2.3. *If $x \in \mu^{-1}(0)$, then $h(x) = (a_1(x) \cdots a_r(x))^{1/4}$.*

Proof. Let v_i, μ_i be as in Subsection 2.1. Let J and $\langle \cdot, \cdot \rangle$ denote the complex structure and the Kähler scalar product on TX. Then

$$\langle v_i, v_j \rangle \ = \ \omega(v_i, Jv_j) \ = \ d\mu_i(Jv_j).$$

Each μ_i is a function on $\mathcal{U}_\varepsilon$ and, hence, may be written as $\mu_i = \sum \alpha_{ij}(x) y_j$.

For any $x \in \mu^{-1}(0)$, consider the $r \times r$-matrices:

$$A(x) \ = \ \big\{ \, \alpha_{ij}(x) \, \big\}, \quad V(x) \ = \ \big\{ \, \langle v_i(x), v_j(x) \rangle \, \big\}, \quad F(x) \ = \ \big\{ \, \langle f_i(x), Jv_j(x) \rangle \, \big\}$$

Clearly,

$$AF \ = \ V, \quad A^2 \ = \ \operatorname{diag}(a_1, \dots, a_r), \quad (\det F)^2 \ = \ \det V \ = \ h^4.$$

Hence, $h^2 = \det V / \det F = \det A = \sqrt{a_1 \cdots a_r}$. □

For any $t > 0$, consider the function β_t on $\mathcal{U}_\varepsilon$ defined by the formula

$$\beta_t(x, Y) \ := \ \Big(\prod_{i=1}^r a_i \Big)^{1/4} \Big(\frac{t}{2\pi} \Big)^{r/4} \exp(-\frac{t}{2} \sum_{i=1}^r a_i y_i^2),$$

where $x \in \mu^{-1}(0), Y \in N_x \simeq \mathbb{R}^r$. Clearly,

$$\int_{\mathbb{R}^r} |\beta_t(x, Y)|^2 \, dy_1 \dots dy_r = 1, \tag{2.6}$$

for any $x \in \mu^{-1}(0)$.

Let $\sigma : X \to [0, 1]$ be a smooth function on X, which is identically equal to 1 on $\mathcal{U}_{\frac{\varepsilon}{2}}$ and such that $\operatorname{supp} \sigma \subset \mathcal{U}_\varepsilon$. We can and we will consider the product $\sigma \frac{\beta_t}{h}$ as a function on X, supported on $\mathcal{U}_\varepsilon$.

Recall, from Subsection 1.3, that $q : X^{ss} \to X_0$ denotes the projection. Set

$$\Psi_t^j \ := \ \sigma \frac{\beta_t}{h} \circ q^* : \Omega^{0,j}(X_0, L_0) \to \Omega^{0,j}(X, L), \qquad j = 0, 1, \dots$$

Here $q^* : \Omega^{0,j}(X_0, L_0) \to \Omega^{0,j}(X^{ss}, L^{ss})$ denotes the pull-back, and we view multiplication by the compactly supported function $\sigma \frac{\beta_t}{h}$ as a map from $\Omega^{0,j}(X^{ss}, L^{ss})$ to $\Omega^{0,j}(X, L)$.

It follows from (2.6) and the definition of h that, for $t \gg 0$, the map Ψ_t^* is closed to isometry, i.e.,

$$\lim_{t \to \infty} \| \Psi_t^* \Psi_t - \operatorname{Id} \| = 0. \tag{2.7}$$

Remark 2.4. Clearly Ψ_t^* does not commute with differentials, due to the presence of the cut-off function σ in the definition of Ψ_t^* and also because $|\mu|^2$ is only approximately equal to $\sum a_i y_i^2$. However, for any $\alpha \in \Omega^{0,*}(X_0, L_0)$, the restriction of $\Psi_t^* \bar{\partial} \alpha$ to $\mathcal{U}_{\frac{\varepsilon}{2}}$ is "very close" to $\bar{\partial}_t \Psi_t^* \alpha$. Since, for large values of t, "most of the norm" of $\Psi_t^* \bar{\partial} \alpha$ is concentrated in $\mathcal{U}_{\frac{\varepsilon}{2}}$, we see that, for $t \gg 0$, the map Ψ_t^* "almost commutes" with differentials. More precisely, $\lim_{t \to \infty} \| \Psi_t^* \bar{\partial} \alpha - \bar{\partial}_t \Psi_t^* \alpha \| = 0$ for any $\alpha \in \Omega^{0,*}(X_0, L_0)$.

Let $\overline{E}_t^j \subset \Omega^{0,j}(X, L)$ denote the image of $\mathcal{H}^j(X_0, L_0)$ under Ψ_t^j.

The following theorem, which combines Theorem 3.10 and Corollary 3.6 of [11] and Theorem 10.1 of [1], shows that the image of $\mathcal{H}^*(X_0, L_0)$ under Ψ_t^* is "asymptotically equal" to $E_{\lambda,t}^{*,K}$.

Theorem 2.5. *Let $P_{\lambda,t}^j : \Omega^{0,j}(X, L) \to E_{\lambda,t}^{j,K}$ be the orthogonal projection and let $\operatorname{Id} : \Omega^{0,j}(X, L) \to \Omega^{0,j}(X, L)$ be the identity operator. Then, there exists $\lambda > 0$, such that*

$$\lim_{t \to \infty} \| (\operatorname{Id} - P_{\lambda,t}^j)|_{\overline{E}_t^j} \| \ = \ 0,$$

where $\| (\operatorname{Id} - P_{\lambda,t}^j)|_{\overline{E}_t^j} \|$ denotes the norm of the restriction of $\operatorname{Id} - P_{\lambda,t}^j$ to $\overline{E}_t^j$.

Define the map

$$\Phi^j_{\lambda,t} : \mathcal{H}^j(X_0, L_0) \ \to \ E^{j,K}_{\lambda,t},$$

by the formula $\Phi^j_{\lambda,t} \overset{\text{def}}{=} P^j_{\lambda,t} \circ \Psi^j_t$. It follows from Theorem 2.5, that, the map $\Phi^j_{\lambda,t}$ is a monomorphism. With a little more work, cf. [11], one proves the following

Theorem 2.6. *There exists $\lambda, t_0 > 0$ such that, for every $j = 0, 1, \dots$ and every $t > t_0$, the map $\Phi^j_{\lambda,t} : \mathcal{H}^j(X_0, L_0) \to E^{j,K}_{\lambda,t}$ is an isomorphism of vector spaces.*

This implies, in particular, Theorem 2.1.

From (2.7) and Theorem 2.5 we also obtain the following

Corollary 2.7. $\lim\limits_{t\to\infty} \|\Phi^*_{\lambda,t} - \Psi^*_t\| = 0.$

3. The integration map. Proof of Theorem 1.4

In this section we construct a map $I_t : \Omega^{0,j}(X, L) \to \Omega^{0,j}(X_0, L_0)$, whose restriction to $E^{*,K}_{\lambda,t}$ is "almost equal" to $(\Phi^*_{\lambda,t})^{-1}$ for $t \gg 0$, and such that $I_t\bar\partial_t = \bar\partial I_t$, for any $t \geq 0$. The very existence of such a map implies (cf. Corollary 3.2) that the restriction of $\bar\partial_t$ onto the space $E^{*,K}_{\lambda,t}$ vanishes. Theorem 1.4 follows then from Theorem 2.1.

3.1. **The integration map.** Recall that $q : X^{ss} \to X_0 = X^{ss}/G$ is a fiber bundle. Recall, also, that we denote $r = \dim_{\mathbb{R}} K = \dim_{\mathbb{C}} G$. Hence, $\dim_{\mathbb{C}} q^{-1}(x) = r$, for any $x \in X_0$. The action of G defines a trivialization of L^{ss} along the fibers of q. Using this trivialization, we define a map $I_t : \Omega^{0,j}(X, L) \to \Omega^{0,j}(X_0, L_0)$ by the formula

$$I_t : \alpha \ \mapsto \ \left(\frac{t}{2\pi}\right)^{r/4} \int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}} \alpha \wedge \omega^r, \tag{3.1}$$

Though the integral is taken over a non-compact manifold $q^{-1}(x)$ it is well defined. Indeed, by [3, §4], $q^{-1}(x)$ is the set of smooth points of a complex analytic submanifold of X. Hence, cf. [2, §0.2], the Liouville volume $\int_{q^{-1}(x)} \omega^r$ of $q^{-1}(x)$ is finite. It follows that the integral in (3.1) converges. Moreover, it follows from (2.5) that there exists a constant $C > 0$ such that

$$\| \, I_t \, \| \ \leq \ C, \tag{3.2}$$

for any $t > 0$.

The following theorem describes the main properties of the integration map I_t.

Theorem 3.1. a. *$\bar\partial \circ I_t = I_t \circ \bar\partial_t$, for any $t \geq 0$.*

b. Let $i : \mathcal{H}^*(X_0, L_0) \to \Omega^{0,*}(X_0, L_0)$ denote the inclusion and let $\|I_t \circ \Phi^*_{\lambda,t} - i\|$ denote the norm of the operator $I_t \circ \Phi^*_{\lambda,t} - i : \mathcal{H}^*(X_0, L_0) \to \Omega^{0,*}(X_0, L_0)$. Then

$$\lim_{t\to\infty} \|I_t \circ \Phi^*_{\lambda,t} - i\| \ = \ 0.$$

We postpone the proof of Theorem 3.1 to the next section. Now we will show how it implies Theorem 1.4 (and, hence, also Theorems 1.1 and 1.3). First, we establish the following simple, but important corollary of Theorem 3.1.

Corollary 3.2. *Recall that a positive number t_0 was defined in Theorem 2.6. Choose $t > t_0$ large enough, so that $\|I_t \circ \Phi^*_{\lambda,t} - i\| < 1$, cf. Theorem 3.1. Then, $\bar\partial_t\gamma = 0$ for any $\gamma \in E^{*,K}_{\lambda,t}$.*

Proof. Let t be as in the statement of the corollary and let $\gamma \in E^{*,K}_{\lambda,t}$. Then, $\bar\partial_t\gamma \in E^{*,K}_{\lambda,t}$. Hence, it follows from Theorem 2.6, that there exists $\alpha \in \mathcal{H}^*(X_0, L_0)$ such that $\Phi^*_{\lambda,t}\alpha = \bar\partial_t\gamma$.

By Theorem 3.1.a, the vector $I_t\bar\partial_t\gamma = \bar\partial I_t\gamma \in \Omega^{0,*}(X_0, L_0)$ is orthogonal to the subspace $\mathcal{H}^*(X_0, L_0)$. Hence,

$$\|\alpha\| \ \le\ \|I_t\bar\partial_t\gamma - \alpha\| \ =\ \|I_t\Phi^*_{\lambda,t}\alpha - \alpha\|.$$

Since, $\|I_t \circ \Phi^*_{\lambda,t} - i\| < 1$, it follows that $\alpha = 0$. Hence, $\bar\partial_t\gamma = \Phi^*_{\lambda,t}\alpha = 0$. □

3.2. **Proof of Theorem 1.4.** We have already mentioned in Subsection 2.2, that $E^{*,K}_{\lambda,t}$ is a subcomplex of $(\Omega^{0,*}(X, L), \bar\partial_t)$, whose cohomology is isomorphic to $H^*(X, \mathcal{O}(L))^K$. Since, by Corollary 3.2, the differential of this complex is equal to 0, we obtain

$$\dim H^j(X, \mathcal{O}(L))^K = \dim E^{j,K}_{\lambda,t}.$$

Theorem 1.4 follows now from Theorem 2.1. □

4. Proof of Theorem 3.1

4.1. **Proof of Theorem 3.1.a.** The first part of Theorem 3.1 is an immediate consequence of the following two lemmas:

Lemma 4.1. $\displaystyle\int_{q^{-1}(x)} \mu_i\bar\partial\mu_i \wedge \alpha \wedge \omega^r \ =\ 0$, *for any* $i, j = 0, 1, \dots$ *and any* $\alpha \in \Omega^{0,j}(X, L)$.

Proof. For any $l = 0, 1, \dots,$ let $\Pi_{0,l} : \Omega^{*,*}(X_0, L_0) \to \Omega^{0,l}(X_0, L_0)$ denote the projection. Then

$$\int_{q^{-1}(x)} \mu_i\bar\partial\mu_i \wedge \alpha \wedge \omega^r \ =\ \Pi_{0,j+1} \int_{q^{-1}(x)} \mu_i d\mu_i \wedge \alpha \wedge \omega^r. \tag{4.1}$$

Using (2.1), we obtain

$$\begin{aligned}\int_{q^{-1}(x)} \mu_i d\mu_i \wedge \alpha \wedge \omega^r \ &=\ \int_{q^{-1}(x)} \mu_i\iota_{v_i}\omega \wedge \alpha \wedge \omega^r \\ &=\ \frac{1}{r+1}\int_{q^{-1}(x)} \mu_i\, \iota_{v_i}(\omega^{r+1} \wedge \alpha) \ -\ \frac{1}{r+1}\int_{q^{-1}(x)} \mu_i\omega^{r+1} \wedge \iota_{v_i}\alpha.\end{aligned} \tag{4.2}$$

The first summand in (4.2) vanishes since the vector v_i is tangent to $q^{-1}(x)$. The integrand in the second summand belongs to $\Omega^{r+1,r+j}(X, L)$. It follows, that

$$\int_{q^{-1}(x)} \mu_i d\mu_i \wedge \alpha \wedge \omega^r \ \in\ \Omega^{1,j}(X_0, L_0).$$

The lemma follows now from (4.1). □

Lemma 4.2. $\displaystyle\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\bar\partial\alpha \wedge \omega^r \ =\ \bar\partial\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\alpha \wedge \omega^r$, *for any* $j = 0, 1\dots$ *and any* $\alpha \in \Omega^{0,j}(X, L)$.

Proof. Since the complement of X^{ss} in X has real codimension ≥ 2, there exists a sequence $\alpha_k \in \Omega^{0,j}(X, L)$, $k = 1, 2\dots$ convergent to α in the topology of the Sobolev space $W^{1,1}$, and such that $\mathrm{supp}(\alpha_k) \subset X^{ss}$. Hence, it is enough to consider the case when support of α is contained in X^{ss}, which we will henceforth assume.

Let $\beta \in \Omega^{n-j,n-j-1}(X_0, L_0^*)$, where n is the complex dimension of X and L_0^* denotes the bundle dual to L_0.

By Lemma 4.1,

$$\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\bar\partial\alpha\wedge\omega^r \;=\; \int_{q^{-1}(x)} \bar\partial\big(e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\big) \\ +\; t\sum_{i=1}^r\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\mu_i\bar\partial\mu_i\wedge\alpha\wedge\omega^r \;=\; \int_{q^{-1}(x)} \bar\partial\big(e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\big).$$

Hence,

$$\int_{X_0}\Big(\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\bar\partial\alpha\wedge\omega^r\Big)\wedge\beta \\ =\; \int_{X^{ss}}\bar\partial\big(e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\big)\wedge q^*\beta \;=\; \int_X\bar\partial\big(e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\big)\wedge q^*\beta \\ =\; (-1)^j\int_X e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\wedge q^*(\bar\partial\beta) \;=\; (-1)^j\int_{X^{ss}} e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\wedge q^*(\bar\partial\beta) \\ =\; (-1)^j\int_{X_0}\Big(\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\Big)\wedge\bar\partial\beta \;=\; \int_{X_0}\bar\partial\Big(\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\alpha\wedge\omega^r\Big)\wedge\beta.$$

□

This completes the proof of Theorem 3.1.a. □

4.2. **Proof of Theorem 3.1.b.** Fix $\alpha\in\mathcal{H}^*(X_0,L_0)$. Using (3.2) and Corollary 2.7, we obtain

$$\|\, I_t\Phi^*_{\lambda,t}\alpha - I_t\Psi^*_t\alpha\,\| \;\le\; C\,\|\,(\Phi^*_{\lambda,t}-\Psi^*_t)\alpha\,\| = o(1),$$

where $o(1)$ denotes a form, whose norm tends to 0, as $t\to\infty$. Hence,

$$I_t\Phi^*_{\lambda,t}\alpha \;=\; I_t\Psi^*_t\alpha \;+\; o(1) \;=\; \Big(\frac{t}{2\pi}\Big)^{r/4}\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\sigma\frac{\beta_t}{h}q^*\alpha\wedge\omega^r \;+\; o(1) \\ =\; \Big(\Big(\frac{t}{2\pi}\Big)^{r/4}\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\sigma\frac{\beta_t}{h}\,\omega^r\Big)\cdot\alpha \;+\; o(1).$$

Recall that in Subsection 2.4, we introduced coordinates in a neighborhood $\mathcal{U}_\varepsilon$ of $\mu^{-1}(0)$. Using this coordinates and the definition of the function h, we can write

$$\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\sigma\frac{\beta_t}{h}\omega^r \;=\; \int_{\mathbb{R}^r} e^{-\frac{t|\mu|^2}{2}}h\sigma\beta_t dy_1\cdots dy_r.$$

Hence, from (2.5) and Lemma 2.3, we obtain

$$\Big(\frac{t}{2\pi}\Big)^{r/4}\int_{q^{-1}(x)} e^{-\frac{t|\mu|^2}{2}}\sigma\frac{\beta_t}{h}\omega^r \\ =\; \Big(\frac{t}{2\pi}\Big)^{r/2}\Big(\prod_{i=1}^r a_i\Big)^{1/2}\int_{\mathbb{R}^r}\sigma\exp\Big(-\frac{t}{2}\sum_{i=1}^r a_iy_i^2\Big)\,dy_1\cdots dy_r + o(1) \\ =\; 1 \;+\; o(1). \qquad (4.3)$$

Thus, $I_t\Phi^*_{\lambda,t}\alpha = \alpha + o(1)$. In other words, the operator $I_t\Phi^*_{\lambda,t}$ converges to i as $t\to\infty$ in the strong operator topology. Since the dimension of $\mathcal{H}^*(X_0,L_0)$ is finite, it also converges in the norm topology. □

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